Approximate Capacitance Expressions for Two Equal Sized Conducting Spheres

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***Abstract*—Charged conducting spheres can be used to model the interaction of objects such as water droplets in clouds. In this paper we provide approximate expressions for the capacitance of two equal sized conducting spheres. These capacitance expressions span the full range of separation (zero to infinity) between the two spheres and are accurate to within 0.1% at all distances. Using these expressions the force and the energy of electrostatic interaction between two conducting spheres can be calculated quickly and accurately.**

I. Introduction

We revisit the classic problem of the electrostatic interaction of two charged conducting spheres [1], [2], [3], [4]. The interaction has applications in any system with charged particles such as granular materials, colloids, aerosols, rain clouds etc. Due to its many applications the problem continues to generate interest to this day [5], [6], [7].

The solution to this problem requires calculating the capacitance coefficients for the two spheres. Using the capacitance coefficients, the electrostatic energy and the mutual force of interaction can be calculated. The classical solution for the capacitance coefficients is in the form of infinite series which converge slowly when the two spheres are to close to each other.

This slow convergence of the capacitance series (and thus the force series) can be a problem in situations where the force needs to be calculated quickly such as computer modeling of the dynamics of particles and droplets in applications discussed above. Approximate analytic expressions for the interaction of the two spheres are thus needed.

We study the simpler case of two equal sized conducting spheres. Approximate expressions for their interaction exist in the near and far limits of the separation of two spheres. However, to our knowledge, no expressions exist that span the full range of their separation. In this paper we provide approximate expressions for the capacitance coefficients of two interacting spheres that not only span their full range of separation but are accurate to within 0.1% at all distances.

II. Classical Theory of Capacitance

Consider two conducting spheres A and B of radii a and b such that a=b with a center to center distance r between them. Let the two spheres be at voltages V_a and V_b respectively as shown in Fig. 1.

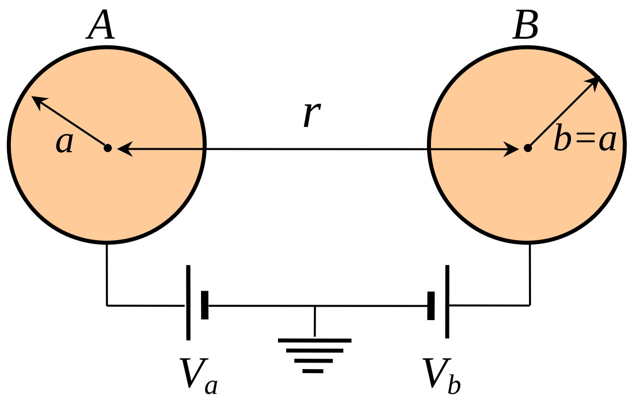

Fig.1. Basic geometry of our setup. The charge on each sphere depends upon the applied voltages V_a and V_b, and the distance r between them.

The coefficients of capacitance for the spheres are defined through to the relation of charges Q_a and Q_b on the spheres to their voltages:

$$\begin{aligned} Q_a &= C_{aa}V_a + C_{ab}V_b, \\ Q_b &= C_{ab}V_a + C_{bb}V_b. \end{aligned} \tag{1}$$

Here C_{aa} = C_{bb} are the self capacitances of the two spheres, and C_{ab} is their mutual capacitance.

These capacitance coefficients are calculated using the method of images to give the well known result [2]

$$\begin{aligned} C_{aa} &= 4\pi\varepsilon_0 a \sinh\beta \sum_{n=1}^{\infty} \operatorname{cosech}[(2n-1)\beta], \\ C_{ab} &= -4\pi\varepsilon_0 a \sinh\beta \sum_{n=1}^{\infty} \operatorname{cosech}[2n\beta], \end{aligned} \tag{2}$$

where $\beta = \cosh^{-1}[r/2a]$.

The infinite series solutions in Equation 2 for the capacitance coefficients were evaluated numerically by Pisler and Adhikari [8] accurately to better than one part in 10^6. As the distance between the spheres decreased they needed higher and higher number of terms to achieve the desired accuracy.

III. Limiting behavior of Capacitance Coefficients

The capacitance coefficients of two equal spheres have been examined in both near and far regimes. Maxwell [1] gives a series expansion to 21 terms in the far limit, $r>>a$. With modern computers the results in Equation 2 can easily be expanded to almost arbitrary number of terms in power of $1/r$. The first few leading terms are given by

$$\begin{aligned} c_{aa} &\equiv \frac{C_{aa}}{4\pi\varepsilon_0 a} = 1 + \frac{a^2}{r^2} + \frac{2a^4}{r^4} + O\left(\frac{a^6}{r^6}\right), \\ c_{ab} &\equiv \frac{C_{ab}}{4\pi\varepsilon_0 a} = -\frac{a}{r} - \frac{a^3}{r^3} - \frac{3a^5}{r^5} - O\left(\frac{a^7}{r^7}\right), \end{aligned} \tag{3}$$

where c_{aa} and c_{ab} are the dimensionless versions of the capacitance coefficients C_{aa} and C_{ab} respectively. The leading term in c_{ab} gives rise to the Coulombic term in the electrostatic force between the spheres. From here on we follow a similar notation to that used by Crowley [9] in discussing the interaction of a sphere with a conducting wall.

In discussing the near limit, $r \to 2a$ it is convenient to define the relative separation between spheres

$$\xi \equiv \frac{r-2a}{2a}. \tag{4}$$

As the spheres approach each other their relative separation, ξ, goes to 0. In this limit the behavior is complicated by the presence of logarithmic singularities and, therefore, the expansions are known only to a few terms [3], [4], [5].

$$\begin{aligned} c_{aa} &= -\frac{1}{4}\log\xi + \frac{\gamma}{2} + \frac{3}{4}\log 2 + O(\xi\log\xi), \\ c_{ab} &= \frac{1}{4}\log\xi - \frac{\gamma}{2} + \frac{1}{4}\log 2 + O(\xi\log\xi). \end{aligned} \tag{5}$$

Here $\gamma = 0.577216\ldots$ is the EulerGamma constant.

In Fig. 2 below we plot the dimensionless capacitance coefficients versus their relative separation on a log-linear plot. The logarithmic divergences of the capacitances in Equation 5 are apparent from the straight line asymptotic behavior in the limit $\xi \to 0$. In the far limit c_{aa} and c_{ab} tend asymptotically to 1 and 0 respectively as expected from Equation 3.

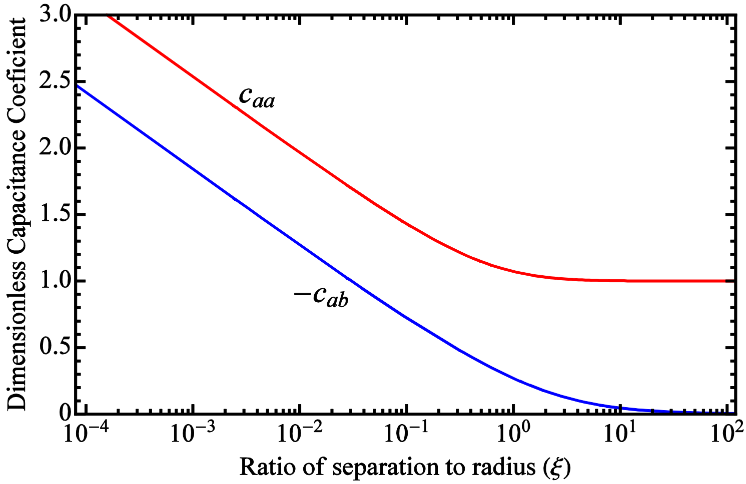

Fig.2. The dimensionless capacitance coefficients for two equal sized sphere are plotted versus their relative separation. At small separation they diverge logarithmically. At large distances c_{aa} goes to 1, the capacitance of an isolated sphere and c_{ab} goes as inverse of the separation.

IV. Approximate Capacitance Coefficients

In this section we present our approximate expressions for the capacitances of two equal spheres which span the whole range of distances between the two spheres. To our knowledge no such expressions exist at the moment in the electrostatics lecture.

We begin with the problem of a conducting sphere interacting with a conducting wall which is a special case of the two sphere problem with the additional condition $V_b = -V_a$. For this special case the capacitance follows from Equation 1

$$c = c_{aa} - c_{ab}. \tag{6}$$

For this sphere and wall problem an approximation for the capacitance c was first derived by Hudlet et al [10] in the context of atomic force microscopy. Crowley [9] later expressed this approximation in the elegant dimensionless form

$$\tilde{c} = 1 + \frac{1}{2}\log 1 + \frac{1}{\xi}. \tag{7}$$

The simple expression in Equation 7 is surprisingly effective; accurate to within about 2% at all distances between the sphere and the wall.

In a manuscript currently submitted for publication [11] we improved upon the approximation in Equation 7 by re-expressing it in terms of variable $x \equiv 1 + \xi - \sqrt{\xi(2+\xi)}$ and adding a correction term. The additional term cancels the leading order of difference between the approximation in Equation 7 and the exact capacitance c defined in Equation 6. Our improved approximation for the capacitance c is

$$\tilde{c}_2 = 1 + \frac{1}{2}\log\frac{1+x^2}{(1-x)^2} + k\left(\frac{2x-x^2}{2-2x+x^2}\right)^{2.15}, \tag{8}$$

with the coefficient of the correction term $k = \gamma + \frac{1}{2}\log 2 - 1 = -0.0762107\ldots$ The additional term improves the accuracy of the approximation by more than one order of magnitude from 2% to about 0.05%.

Using limiting analysis from Section III and insights gained from Equation 8 we postulate the following approximations for the two equal sphere case:

$$\begin{aligned}\tilde{c}_{ab} &= -\frac{x}{1+x^2}\left(1 + \frac{1}{2}\log\frac{1+x^4}{(1-x^2)^2} + k\left(\frac{2x^2-x^4}{2-2x^2+x^4}\right)^{2.15}\right), \\ \tilde{c}_{aa} &= \tilde{c}_2 + \tilde{c}_{ab}.\end{aligned} \tag{10}$$

These approximations in Equation 10 are much easier for a computer to calculate than the infinite series in Equation 2. These approximations span the full range of separation between the spheres and are highly accurate. In Fig. 3 below we plot the percentage error in on a log-linear plot. The

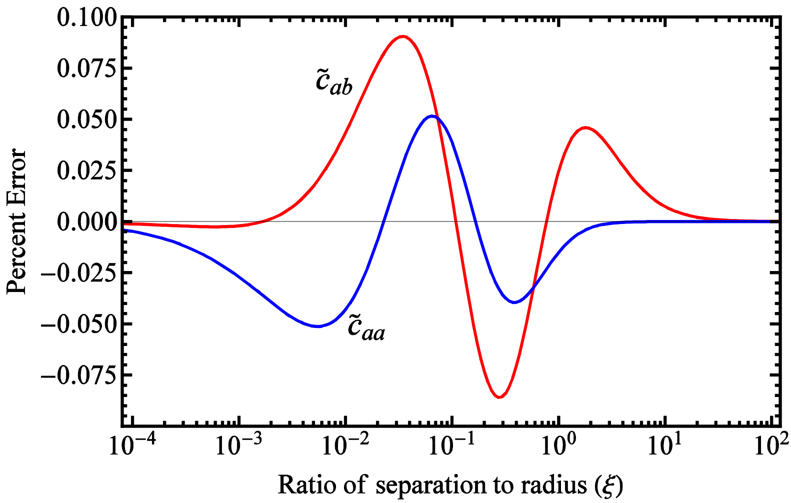

Fig. 3. The percentage error in our approximate capacitance expressions is plotted versus the relative separation of spheres. The maximum error for c_{ab} is within 0.1% and for c_{aa} is within 0.05% at all distances. In the near and far limits of relative separation the error goes to zero.

maximum error in the approximations are below 0.1%. Unlike any other approximations available in the literature, the expressions in Equation 10 span the full range of separation between the spheres.

V. CONCLUSION

We provide approximate capacitance expressions for the interaction of two equal sized conducting spheres that are accurate to within 0.1% and span the whole range of separation between the spheres. From these approximations the electrostatic potential energy and the electrostatic force between the spheres can be calculated using methods outlined in electrostatics books such as [1] and [2]. Our approximations can be a useful tool for scientists and engineers who wish to calculate the interaction between the spheres quickly and accurately avoiding evaluation of expressions with infinite series.

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